Critical Galton-Watson processes: The maximum of total progenies within a large window

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Abstract

Consider a critical Galton-Watson process $Z = \{Z_n : n = 0, 1, \ldots\}$ of index $1+\alpha,\ \alpha \in (0,1]$. Let $S_k(j)$ denote the sum of the Z_n with n in the window $[k, \ldots, k+j)$, and $M_m(j)$ the maximum of the $S_k(j)$ with k moving in $[0, m-j]$. We describe the asymptotic behavior of the expectation $\mathbf{E}M_m(j)$ if the window width $j = j_m$ is such that j/m converges in $[0,1]$ as $m \uparrow \infty$. This will be achieved via establishing the asymptotic behavior of the tail probabilities of $M_\infty(j)$.

Contents

1 Introduction and statement of results

Let $Z = \{Z_n : n \geq 0\}$ denote a Galton-Watson process. As a rule, we start with a single ancestor: $Z_0 = 1$. It will be convenient to write ξ for the intrinsic number of offspring Z_1. We always assume that Z is *critical*, that is $\mathbf{E}\xi = 1$. If not stated otherwise, we consider the case of *branching of index* $1+\alpha$ for some $0 < \alpha \leq 1$. With this we mean that the related offspring generating function f satisfies

$$f(s) := \mathbf{E}s^{\xi} = s + (1-s)^{1+\alpha} L(1-s), \qquad 0 \leq s \leq 1, \tag{1}$$

where $x \mapsto L(x)$ is a function slowly varying as $x \downarrow 0$. For $k \geq 0$ and $1 \leq j \leq m < \infty$, set

$$S_k(j) := \sum_{l=k}^{k+j-1} Z_l \quad \text{and} \quad M_m(j) := \max_{0 \leq k \leq m-j} S_k(j). \tag{2}$$

Extend these notations by monotone convergence to $m = \infty$ or even $j = \infty$, and put

$$M(j) := M_\infty(j), \qquad 1 \leq j \leq \infty. \tag{3}$$

Since any critical Galton-Watson process dies a.s. in finite time, $M(j)$ is a proper random variable for any j. In particular, $M(\infty)$ coincides with the total number $S_0(\infty) = Z_0 + Z_1 + \cdots$ of individuals of Z.

The *main purpose* of this note is to study the asymptotic behavior of the expectation $\mathbf{E}M_m(j)$ when j might depend on m such that $j/m \to \eta \in [0,1]$ as $m \uparrow \infty$.

To get a feeling, let us first discuss two special cases. If $j = m$, we have

$$\mathbf{E}M_m(m) \,=\, \mathbf{E}S_0(m) \,=\, \mathbf{E}\sum_{l=0}^{m-1} Z_l \,=\, m. \tag{4}$$

On the other hand, the case $j = 1$ reduces to the investigation of the asymptotic behavior of the expectation of $M_m(1) =: M_m = \max_{0\le k\le m-1} Z_k$ as $m \uparrow \infty$. The last issue has a rather long history. First Weiner [Wei84] demonstrated that if the critical process has a finite variance [which requires $\alpha = 1$ in our case (1)], then there exist constants $0 < \underline{c} \le \bar{c} < \infty$ such that $\underline{c} \le \mathbf{E}M_m/\log m \le \bar{c}$ for all m. Then Kämmerle and Schuh [KS86] and Pakes [Pak87] have found explicit bounds for $\underline{c}$ from below and for $\bar{c}$ from above. Finally, Athreya [Ath88] established (still under the condition $\mathbf{V}\mathrm{ar}\xi < \infty$) that

$$\mathbf{E}M_m(1) \,=\, \mathbf{E}M_m \,\sim\, \log m \quad \text{as } m \uparrow \infty. \tag{5}$$

In Borovkov and Vatutin [BV96] the validity of (5) was proved under condition (1). Moreover, in Vatutin and Topchii [VT97] and Bondarenko and Topchii [BT01] asymptotics (5) was established under much weaker conditions than (1), for instance, in [BT01] under $\mathbf{E}\xi \log^{\beta}(1+\xi) < \infty$ for any $\beta > 0$.

Comparing the difference of orders at the right-hand sides of (4) and (5) leads to the following natural question: What can be said about the behavior of $\mathbf{E}M_m(j)$ when the width j of the moving window within which the total population size is calculated, may vary anyhow with m. For this purpose, we restrict our attention to processes satisfying (1). Here is our *main result.*

Theorem 1 (Expected maximal total progeny) *Assume that* $j_m \ge 1$ *satisfies* $j_m/m \to \eta \in [0,1]$ *as* $m \uparrow \infty$.

(a) *If* $\eta = 0$, *then*

$$\mathbf{E}M_m(j_m) \,\sim\, j_m \log\Big(\frac{m}{j_m}\Big) \quad \textit{as } m \uparrow \infty. \tag{6}$$

(b) *If* $0 < \eta \le 1$, *then*

$$\mathbf{E}M_m(j_m) \,\sim\, j_m \varphi(\eta) \quad \textit{as } m \uparrow \infty, \tag{7}$$

where φ *is explicitly given in formula* (177) *below. In particular,*

$$\varphi(\eta) \,\sim\, \log\frac{1}{\eta} \quad \textit{as } \eta \downarrow 0. \tag{8}$$

Note that (8) yields a continuous transition between the cases (a) and (b).

We will deduce Theorem 1 via studying finer properties of $M(j) = M_\infty(j)$. In fact, we will establish the following asymptotic representation for tail probabilities of $M(j)$. As usual, we write $Q(n)$ for the survival probability $\mathbf{P}(Z_n > 0)$.

Theorem 2 (Tail of maximal total progeny) *Assume that $j_n \geq 1$ satisfies $j_n/\bigl(Q(j_n)n\bigr) \to y \in [0,\infty]$ as $n \uparrow \infty$.*

(a) *If $y = \infty$, then*

$$\mathbf{P}\bigl(M(j_n) \geq n\bigr) \sim \mathbf{P}\bigl(M(\infty) \geq n\bigr) \sim n^{-\frac{1}{1+\alpha}}\,\ell(n) \quad as \ n \uparrow \infty, \tag{9}$$

where ℓ is a function slowly varying at infinity.

(b) *If $0 < y < \infty$, then*

$$\mathbf{P}\bigl(M(j_n) \geq n\bigr) \sim Q(j_n)\,\psi(y) \quad as \ n \uparrow \infty, \tag{10}$$

where ψ is explicitly given in formula (139) *below.*

(c) *Finally, if $y = 0$, then*

$$\mathbf{P}\bigl(M(j_n) \geq n\bigr) \sim \mathbf{P}\bigl(M(1) \geq n j_n^{-1}\bigr) \sim \frac{\alpha j_n}{n} \quad as \ n \uparrow \infty. \tag{11}$$

The rest of the paper is organized as follows. In the next two subsections, we state some (partially known) properties of critical Galton-Watson processes, preparing for the proof of parts (a) and (c) of Theorem 2, given in Subsection 3.1. This is followed in 2.3 by a conditional invariance principle for critical Galton-Watson processes of index $1+\alpha$, see Proposition 13, needed for the proof of Theorem 2(b) (also given in 3.1). Properties of the limit process X^* arising in the mentioned invariance principle, are studied in 2.4 and applied as Proposition 14 in the proof of Theorem 1(b) in Subsection 3.3.

2 Auxiliary tools

2.1 Basic properties of critical processes of index $1+\alpha$

We start with some further notational conventions. If symbols L and ℓ [as in (1) and (9), respectively] have an index, they also denote functions slowly varying at zero or infinity, respectively. In this case, the index might refer to the first place of its occurrence, for instance, $\ell_{\#}$ for occurring in Lemma #. Furthermore, the letter c will always denote a (positive and finite) constant, which might change from place to place, except it has an index, which also might refer to the place of first occurrence. We will also use the following

convention: If a mathematical expression (as Z_n) is defined only for an integer (here n), but we write a non-negative number in it instead (as Z_x), then actually we mean the integer part of that number (here $Z_{[x]}$).

Now we collect some basic properties of critical processes under our assumption (1). The first lemma is taken from Vatutin [Vat81, Lemma 1].

Lemma 3 (Asymptotics of f') *For f from* (1) *we have*

$$1-f'(s)\;\sim\;(1+\alpha)(1-s)^{\alpha}L(1-s)\quad\text{as }\; s\uparrow 1. \tag{12}$$

The next lemma is due to Slack [Sla68].

Lemma 4 (Asymptotics of survival probability) *As $n\uparrow\infty$,*

$$\alpha\, Q^{\alpha}(n)\, L\bigl(Q(n)\bigr)\;\sim\;1/n, \tag{13}$$

implying

$$Q(n)\;\sim\;n^{-1/\alpha}\,\ell_4(n) \tag{14}$$

for a function ℓ_4 (slowly varying at infinity).

Set $f_0(s):=s,$ and $f_n(s):=f(f_{n-1}(s)),\ n\geq 1,$ for the iterations of f. The following lemma can be considered as a local limit statement.

Lemma 5 (A local limit statement) *As $n\uparrow\infty$,*

$$d_n\;:=\;\prod_{k=1}^{n-1}f'\bigl(f_k(0)\bigr)\;\sim\;n^{-1-1/\alpha}\,\ell_5(n). \tag{15}$$

Proof It follows from Lemmas 3 and 4 that

$$1-f'\bigl(1-Q(k)\bigr)\;=\;\frac{1+\alpha}{\alpha k}\,\bigl(1+\delta(k)\bigr), \tag{16}$$

where $\delta(k)\to 0$ as $k\uparrow\infty$. Hence,

$$\begin{aligned} d_n\;&=\;\exp\Bigl[\sum_{k=1}^{n-1}\log f'\bigl(f_k(0)\bigr)\Bigr]\;=\;\exp\Bigl[-\frac{1+\alpha}{\alpha}\sum_{k=1}^{n-1}\frac{1}{k}\,\bigl(1+\delta_1(k)\bigr)\Bigr]\\ &\sim\;n^{-\frac{1+\alpha}{\alpha}}\,\mathrm{e}^{-\frac{1+\alpha}{\alpha}\gamma}\exp\Bigl[-\frac{1+\alpha}{\alpha}\sum_{k=1}^{n-1}\frac{\delta_1(k)}{k}\Bigr]\quad\text{as }\; n\uparrow\infty, \end{aligned} \tag{17}$$

where γ is Euler's constant, and also $\delta_1(k)\to 0$ as $k\uparrow\infty$. According to Seneta [Sen76, Theorem 1.2 in Section 1.5], the function

$$n\;\mapsto\;\exp\Bigl[-\frac{1+\alpha}{\alpha}\sum_{k=1}^{n-1}\frac{\delta_1(k)}{k}\Bigr] \tag{18}$$

is slowly varying at infinite. Combining this with (17) proves the lemma.□

The next statement might also be known from the literature. Recall that $M(\infty)=S_0(\infty)$.

Lemma 6 (Maximal total population) *As* $n \uparrow \infty$,

$$\mathbf{P}\big(M(\infty) \geq n\big) \;=\; \mathbf{P}\big(S_0(\infty) \geq n\big) \;\sim\; n^{-\frac{1}{1+\alpha}}\, \ell_6(n). \tag{19}$$

Proof As well-known (see, for instance, Harris [Har63, formula (1.13.3)]), $h(s) := \mathbf{E}s^{S_0(\infty)}$ solves the equation

$$h(s) \;=\; s\, f\big(h(s)\big), \qquad 0 \leq s \leq 1. \tag{20}$$

By assumption (1) we have

$$h(s) \;=\; s\, h(s) \;+\; \big(1-h(s)\big)^{1+\alpha}\, L\big(1-h(s)\big), \tag{21}$$

giving in view of $h(1)=1$,

$$\big(1-h(s)\big)^{1+\alpha}\; L\big(1-h(s)\big) \;\sim\; (1-s) \quad \text{as } s \uparrow 1. \tag{22}$$

Hence (cf. [Sen76, Section 1.5]),

$$1-h(s) \;\sim\; (1-s)^{\frac{1}{1+\alpha}}\, L_{(23)}(1-s) \quad \text{as } s \uparrow 1, \tag{23}$$

and

$$\frac{1-h(s)}{1-s} \;=\; \sum_{n=0}^{\infty} \mathbf{P}\big(S_0(\infty) > n\big)\, s^n \;\sim\; \frac{L_{(23)}(1-s)}{(1-s)^{\frac{\alpha}{1+\alpha}}} \quad \text{as } s \uparrow 1, \tag{24}$$

implying (see, for instance, Feller [Fel71, Section XIII]),

$$\mathbf{P}\big(S_0(\infty) \geq n\big) \;\sim\; \frac{1}{\Gamma\big(\frac{\alpha}{1+\alpha}\big)}\, n^{-\frac{1}{1+\alpha}}\, L_{(23)}(1/n) \;=:\; n^{-\frac{1}{1+\alpha}}\, \ell_6(n) \tag{25}$$

as $n \uparrow \infty$. This finishes the proof. □

Lemma 7 (Some tail asymptotics) *The following statements hold.*

(a) *As* $n \uparrow \infty$, *if* $j_n \geq 1$ *satisfies* $j_n/\big(Q(j_n)n\big) \to \infty$, *then*

$$\frac{Q(j_n)}{\mathbf{P}\big(M(\infty) \geq n\big)} \;\to\; 0. \tag{26}$$

(b) *As* $j \uparrow \infty$,

$$\mathbf{P}\big(M(\infty) \geq j/Q(j)\big) \;\sim\; \frac{\alpha^{\frac{1}{1+\alpha}}}{\Gamma\big(\frac{\alpha}{1+\alpha}\big)}\, Q(j). \tag{27}$$

Proof **(a)** Recalling notation h introduced in the beginning of the proof of Lemma 6, set $b_x := 1 - h(1-1/x)$, $x \in [1,\infty)$. As $x \uparrow \infty$, it follows from (22) that

$$b_x^{1+\alpha} L\,(b_x) \;\sim\; 1/x, \tag{28}$$

and from (23) that

$$b_x \;\sim\; x^{-\frac{1}{1+\alpha}} L_{(23)}(1/x). \tag{29}$$

By our assumption, $j_n \to \infty$, hence, by Lemma 4,

$$Q^{1+\alpha}(j_n)\, L\bigl(Q(j_n)\bigr) \sim \frac{Q(j_n)}{\alpha\, j_n} \quad \text{as } n \uparrow \infty. \tag{30}$$

Thus, combined with (28),

$$\frac{Q^{1+\alpha}(j_n)\, L(Q(j_n))}{b_n^{1+\alpha} L\,(b_n)} \;\sim\; \frac{n\, Q(j_n)}{\alpha\, j_n} \;\to 0 \quad \text{as } n \uparrow \infty. \tag{31}$$

Note that the function $s \mapsto (1-s)^{1+\alpha} L\,(1-s) = f(s) - s$ is monotone (its derivative $f'(s) - 1$ is negative for $s \in [0,1)$ by criticality). Applying this to $1 - s = Q(j_n)$ and $1 - s = b_n$, it follows from (31) and properties of slowly varying functions that

$$Q(j_n)/b_n \to 0 \quad \text{as } n\, j_n^{-1} Q(j_n) \to 0. \tag{32}$$

On the other hand, from (25) and (29) it follows that

$$\mathbf{P}\bigl(M(\infty) \geq n\bigr) \;\sim\; \frac{1}{\Gamma\bigl(\frac{\alpha}{1+\alpha}\bigr)}\, b_n \quad \text{as } n \uparrow \infty. \tag{33}$$

Combining this with (32) proves part **(a)** of the lemma.

(b) Observe that by (29) and (33),

$$\mathbf{P}\bigl(M(\infty) \geq j/Q(j)\bigr) \;\sim\; \frac{1}{\Gamma\bigl(\frac{\alpha}{1+\alpha}\bigr)}\, b_{j/Q(j)} \quad \text{as } j \uparrow \infty, \tag{34}$$

and that

$$b_{j/Q(j)}^{1+\alpha}\, L\bigl(b_{j/Q(j)}\bigr) \;\sim\; \frac{Q(j)}{j} \quad \text{as } j \uparrow \infty. \tag{35}$$

This, combined with (30) and properties of slowly varying functions, implies

$$b_{j/Q(j)} \sim \alpha^{\frac{1}{1+\alpha}}\, Q(j) \quad \text{as } j \uparrow \infty. \tag{36}$$

Substituting (36) into (34) finishes the proof. $\square$

2.2 Basic properties of critical processes

For a while, we now discuss *general* critical Galton-Watson processes [i.e. we drop restriction (1)]. For $R \geq 2$, put

$$B_R := \mathbf{E}\bigl\{\xi(\xi-1);\, \xi \leq R\bigr\} \;\text{ and }\; R_0 := \min\{R \geq 2 :\, B_R > 0\} < \infty, \tag{37}$$

and set

$$\mathcal{F}(s) \;:=\; \frac{1-f(s)}{1-s} \;=\; \sum_{j=0}^{\infty} \mathbf{P}\,(\xi > j)\, s^j, \qquad 0 \leq s < 1. \tag{38}$$

Lemma 8 (Truncated variance) *There exists a positive constant* c_8 *such that for any critical Galton-Watson process,*

$$B_R \;\le\; 2\sum_{1\le j\le R} j\,\mathbf{P}(\xi>j) \;\le\; c_8\, R\bigl(1-\mathcal{F}(1-1/R)\bigr), \qquad R\ge 2. \tag{39}$$

Proof The first inequality in (39) essentially follows by integration by parts. For j satisfying $1\le j\le R$ and $x\in(0,1)$, we have the following elementary inequality:

$$1-(1-x)^j \;\ge\; jx\,(1-x)^{j-1} \;\ge\; jx\,(1-x)^R, \tag{40}$$

which can be rewritten as

$$j \;\le\; x^{-1}(1-x)^{-R}\left(1-(1-x)^j\right). \tag{41}$$

Choosing $x=1/R$ and using criticality $\sum_{j=0}^{\infty}\mathbf{P}(\xi>j)=1,$ we get from the first inequality in (39),

$$\begin{aligned} B_R \;&\le\; 2R\Bigl(1-\frac{1}{R}\Bigr)^{-R}\sum_{0\le j\le R}\Bigl(1-\Bigl(1-\frac{1}{R}\Bigr)^j\Bigr)\mathbf{P}(\xi>j)\\ &=\; 2R\Bigl(1-\frac{1}{R}\Bigr)^{-R}\Bigl(1-\sum_{j>R}\mathbf{P}(\xi>j)-\sum_{0\le j\le R}\Bigl(1-\frac{1}{R}\Bigr)^j\mathbf{P}(\xi>j)\Bigr)\\ &\le\; 2R\Bigl(1-\frac{1}{R}\Bigr)^{-R}\Bigl(1-\sum_{j\ge 0}\Bigl(1-\frac{1}{R}\Bigr)^j\mathbf{P}(\xi>j)\Bigr)\\ &\le\; c\,R\bigl(1-\mathcal{F}(1-1/R)\bigr), \end{aligned} \tag{42}$$

as desired. □

The next statement is a particular case of Nagaev and Wachtel [NW05, Theorem 3].

Lemma 9 (A tail estimate) *For* $m\ge 0,\;k\ge 1,\;y_0>0,$ *and* $R\ge 2,$

$$\begin{aligned}\mathbf{P}(M_{m+1}\ge k) \;&\le\; \bigl(y_0+\frac{1}{R}\bigr)\left[\Bigl(1+\frac{1}{1/y_0+(\mathrm{e}^2+\mathrm{e}^{y_0R})mB_R/2}\Bigr)^k-1\right]^{-1}\\ &\quad+\; m\,\mathbf{P}(\xi>R).\end{aligned} \tag{43}$$

If the variance of ξ, for the moment denoted by B_∞, is finite and positive, then by Doob's inequality,

$$\mathbf{P}(M_{m+1}\ge k) \;\le\; \frac{mB_\infty+1}{k^2} \;\le\; (1+1/B_\infty)\,\frac{mB_\infty}{k^2}\,. \tag{44}$$

Estimate (43) allows us to derive an analogous bound without imposing the finiteness of $\mathbf{V}\mathrm{ar}\xi$:

Lemma 10 (A further tail estimate) *There exist finite constants* $c_{(45)}$ *and* c_{10} *such that*

$$\mathbf{P}(M_{m+1} \geq k) \;\leq\; c_{(45)} \frac{mB_k}{k^2} \;+\; m\,\mathbf{P}(\xi > k/2) \tag{45}$$

for all $k, m \geq 1$ *satisfying* $k/(mB_k) > c_{10}$.

We see that, for k sufficiently large, the first term at the right hand side of (45) coincides with (44) concerning the truncated variance B_k (except the choice of the constant). The second term compensates the truncation.

Proof of Lemma 10 In view of Lemma 8, $B_k/k \to 0$ as $k \uparrow \infty$. Hence, there is a constant $c_{(46)} \geq \mathrm{e}$ such that for $k, m \geq 1$ with $k/(mB_k) > c_{(46)}$,

$$\begin{aligned} y_0 \;&:=\; y_0(k,m) \;:=\; \frac{2}{k}\,\log\frac{k}{mB_k} - \frac{3}{k}\log\log\frac{k}{mB_k} \\ &=\; \frac{1}{k}\,\log\Big(\Big(\frac{k}{mB_k}\Big)^2\log^{-3}\Big(\frac{k}{mB_k}\Big)\Big) \;>\; 0. \end{aligned} \tag{46}$$

Hence, letting $R = k/2 \geq 2$ in (43) and observing that B_R is non-decreasing in R, we get from Lemma 9 and our choice of R,

$$\begin{aligned} \mathbf{P}(M_{m+1} \geq k) \;\leq\; \big(y_0 + \frac{2}{k}\big)\Big[\Big(1 + \frac{1}{1/y_0 + (\mathrm{e}^2 + \mathrm{e}^{y_0 k/2})mB_k/2}\Big)^k - 1\Big]^{-1} \\ + m\,\mathbf{P}(\xi > k/2). \end{aligned} \tag{47}$$

From the estimates

$$y_0 \;\leq\; \frac{2}{k}\log\frac{k}{mB_k} \;\leq\; \frac{2}{k}\log\frac{k}{B_{R_0}} \tag{48}$$

being valid for all $k \geq R_0$, it follows that $y_0 = y_0(k,m) \downarrow 0$ as $k \uparrow \infty$, and, in addition, there exists a constant $c_{(49)}$ such that for $k, m \geq 1$ satisfying $k/(mB_k) \geq c_{(49)}$,

$$\begin{aligned} \big(\mathrm{e}^2 + \mathrm{e}^{y_0 k/2}\big)mB_k/2 \;&=\; \Big(\mathrm{e}^2 + \frac{k}{mB_k}\log^{-3/2}\Big(\frac{k}{mB_k}\Big)\Big)mB_k/2 \\ &\leq\; 2k\log^{-3/2}\Big(\frac{k}{mB_k}\Big) \;\leq\; \frac{1}{4y_0}\,. \end{aligned} \tag{49}$$

Hence, for these k, m,

$$1/y_0 + (\mathrm{e}^2 + \mathrm{e}^{y_0 k/2})mB_k/2 \;\leq\; \frac{5}{4y_0}\,. \tag{50}$$

Clearly, for sufficiently small $y_0 > 0$,

$$\begin{aligned} \Big(1 + \frac{1}{1/y_0 + (\mathrm{e}^2 + \mathrm{e}^{y_0 k/2})mB_k/2}\Big)^k \;&\geq\; \Big(1 + \frac{4y_0}{5}\Big)^k \\ =\; \exp\Big[k\log\Big(1 + \frac{4y_0}{5}\Big)\Big] &\geq \exp\Big[\frac{4ky_0}{5}\,\Big(1 - \frac{y_0}{2}\Big)\Big]. \end{aligned} \tag{51}$$

By the definition of y_0 there exists $c_{(52)}$ such that

$$1 - \frac{y_0}{2} \geq \frac{5}{6} \quad \text{and} \quad y_0 > \frac{6}{7}\frac{2}{k}\log\frac{k}{mB_k} \tag{52}$$

for $k/(mB_k) > c_{(52)}$. Thus, we get the bound

$$\Big(1 + \frac{1}{1/y_0 + (\mathrm{e}^2 + \mathrm{e}^{y_0 k/2})mB_k/2}\Big)^k \geq \Big(\frac{k}{mB_k}\Big)^{8/7} \tag{53}$$

for $k/(mB_k) \geq c_{(53)} := \max(c_{(46)}, c_{(49)}, c_{(52)})$. Moreover, if $k/(mB_k) > c_{10} := \max(c_{(53)}, 2)$, then

$$\bigg(\Big(1 + \frac{1}{1/y_0 + (\mathrm{e}^2 + \mathrm{e}^{y_0 k/2})mB_k/2}\Big)^k - 1\bigg)^{-1} \leq 2\Big(\frac{mB_k}{k}\Big)^{8/7}. \tag{54}$$

Combining (46) – (54) gives, for $k/(mB_k) > c_{10}$,

$$\mathbf{P}(M_{m+1} \geq k) \;\leq\; \frac{2}{k}\bigg(2 + \log\Big(\frac{k}{mB_k}\Big)\bigg)\Big(\frac{mB_k}{k}\Big)^{8/7} + m\,\mathbf{P}(\xi > k/2). \tag{55}$$

The boundedness of the function $x \mapsto x^{-1/7}\log x$ for $x \geq 2$ completes the proof of the lemma. □

Now we return to critical processes of index $1+\alpha$.

Lemma 11 (A moment estimate) *Under condition* (1)*, for* $\beta \in (1, 1+\alpha)$*, there is a constant* $c_{11} = c_{11}(\beta)$ *such that*

$$\mathbf{E}Z_m^\beta \;\leq\; c_{11}\,Q^{1-\beta}(m), \qquad m \geq 1. \tag{56}$$

Proof According to assumption (1), for $0 \leq s < 1$,

$$\frac{1 - \mathcal{F}(s)}{1-s} \;=\; \frac{f(s) - s}{(1-s)^2} \;=\; \sum_{l=0}^{\infty} s^l \sum_{i=l+1}^{\infty} \mathbf{P}(\xi > i) \;=\; \frac{L(1-s)}{(1-s)^{1-\alpha}}. \tag{57}$$

Therefore, by Lemma 8, for all sufficiently large k,

$$B_k \;\leq\; c_8\, k^{1-\alpha} L(1/k). \tag{58}$$

On the other hand, formula (57) and a Tauberian theorem (cf. [Fel71, Theorem 13.5.5]) imply

$$\sum_{l=0}^{k-1}\sum_{i=l+1}^{\infty} \mathbf{P}(\xi > i) \;\sim\; \frac{1}{\Gamma(\alpha)}\, k^{1-\alpha} L(1/k) \quad \text{as } k \uparrow \infty. \tag{59}$$

Hence, for sufficiently large k,

$$\sum_{i=k}^{\infty} \mathbf{P}(\xi > i) \;\leq\; \frac{2}{\Gamma(\alpha)}\, k^{-\alpha} L(1/k) \tag{60}$$

and

$$k\,\mathbf{P}(\xi > 2k) \;\le\; \sum_{i=k+1}^{2k} \mathbf{P}(\xi > i) \;\le\; \sum_{i=k}^{\infty} \mathbf{P}(\xi > i), \tag{61}$$

leading to

$$\mathbf{P}(\xi > k) \;\le\; c\,k^{-\alpha-1} L(1/k). \tag{62}$$

Combining (45), (58), and (62), we see that there exist constants $c_{(63)}$ and $c'_{(63)}$ such that, for $m \ge 1$ and all $k > c_{(63)}/Q(m)$,

$$\mathbf{P}(Z_m \ge k) \;\le\; \mathbf{P}(M_{m+1} \ge k) \;\le\; c'_{(63)}\, m\, k^{-1-\alpha} L(1/k). \tag{63}$$

Clearly, for $\beta \in (1, 1+\alpha)$,

$$\mathbf{E} Z_m^{\beta} \;\le\; \sum_{k=0}^{\infty} \beta\, k^{\beta-1} \mathbf{P}(Z_m \ge k). \tag{64}$$

In the range of the latter summation we distinguish between $k \le c_{(63)}/Q(m)$ and $k > c_{(63)}/Q(m)$. Then, by criticality, the sum restricted to the first case is bounded from above by $\beta\, c_{(63)}^{\beta-1} Q^{1-\beta}(m) = c\, Q^{1-\beta}(m)$ (with a constant c depending from β). On the other hand, by (63), the remaining restricted sum is bounded from above by

$$\beta\, c'_{(63)} m \sum_{k > c_{(63)}/Q(m)} k^{\beta-\alpha-2} L(1/k) \;\le\; c\, m\, Q^{1+\alpha-\beta}(m)\, L\big(Q(m)\big) \tag{65}$$

(cf. [Fel71, Theorem 8.9.1]), which by (13) leads also to $c\, Q^{1-\beta}(m)$, finishing the proof. □

Lemma 12 (Lower deviation probabilities) *Fix* $1 < \beta < 1+\alpha$. *Under condition* (1), *for* $\varepsilon > 0$ *there exists a constant* $c_{12} = c_{12}(\beta, \varepsilon)$ *such that for* $j \ge 1$ *and all* y *satisfying* $y \ge 2/\varepsilon$,

$$\mathbf{P}\Big\{ \min_{l<j} Z_l < y \;\Big|\; Z_0 = (1+\varepsilon) y \Big\} \;\le\; c_{12} \Big(\frac{1}{y\, Q(j)} \Big)^{\beta-1}. \tag{66}$$

Moreover, for all j *and* y *satisfying* $y j^{-1} \ge 2/\varepsilon$,

$$\mathbf{P}\Big\{ \sum_{l=0}^{j-1} Z_l < y \;\Big|\; Z_0 = (1+\varepsilon) y j^{-1} \Big\} \;\le\; c_{12} \Big(\frac{j}{y\, Q(j)} \Big)^{\beta-1}. \tag{67}$$

Proof Fix $j \ge 1$ and $y \ge 2/\varepsilon$. Clearly,

$$\begin{aligned} &\mathbf{P}\Big\{ \min_{l \le j-1} Z_l < y \;\Big|\; Z_0 = (1+\varepsilon) y \Big\} \\ &= \mathbf{P}\Big\{ \min_{l \le j-1} (Z_l - Z_0) < y - Z_0 \;\Big|\; Z_0 = (1+\varepsilon) y \Big\}. \end{aligned} \tag{68}$$

Obviously, $y \geq 2/\varepsilon$ implies that $Z_0 - y = \big[(1+\varepsilon)y\big] - y \geq \frac{\varepsilon}{2}\,y$. Therefore (68) is bounded from above by

$$\mathbf{P}\Big\{\max_{l\leq j-1}|Z_l - Z_0| > \frac{\varepsilon}{2}\,y \;\Big|\; Z_0 = (1+\varepsilon)y\Big\}. \tag{69}$$

Using this, Doob's inequality gives

$$\begin{aligned}&\mathbf{P}\Big\{\min_{l\leq j-1} Z_l < y \;\Big|\; Z_0 = (1+\varepsilon)y\Big\}\\ &\leq \Big(\frac{2}{\varepsilon}\Big)^{\beta}\;\frac{\mathbf{E}\Big\{|Z_{j-1}-Z_0|^{\beta}\;\big|\;Z_0=(1+\varepsilon)y\Big\}}{y^{\beta}}.\end{aligned} \tag{70}$$

For the fixed j, let $Z_{j-1}^{(k)}$, $k \geq 1$, denote independent copies of Z_{j-1} given $Z_0 = 1$. Then, by the von Bahr-Esseen inequality [vBE65],

$$\begin{aligned}\mathbf{E}\Big\{|Z_{j-1}-Z_0|^{\beta}\;\big|\;Z_0=(1+\varepsilon)y\Big\} &= \mathbf{E}\Big|\sum_{k=1}^{(1+\varepsilon)y}\big(Z_{j-1}^{(k)}-1\big)\Big|^{\beta}\\ &\leq (1+\varepsilon)y\,\mathbf{E}\big\{|Z_{j-1}-1|^{\beta}\;\big|\;Z_0=1\big\}.\end{aligned} \tag{71}$$

Using now Lemma 11 we see that

$$\mathbf{E}\big\{|Z_{j-1}-1|^{\beta}\;\big|\;Z_0=1\big\} \;\leq\; 1+c_{11}\,Q^{1-\beta}(j) \;\leq\; (1+c_{11})\,Q^{1-\beta}(j). \tag{72}$$

Combining (70) – (72), we obtain (66).

Noting that $\sum_{l=0}^{j-1} Z_l < y$ implies $\min_{l\leq j-1} Z_l < yj^{-1}$, and using verified (66), claim (67) follows, and the proof is finished. □

2.3 A conditional invariance principle

From now on we always impose our basic assumption (1). In this section, we establish convergence in law of the conditional scaled Galton-Watson processes

$$\big\{Q(n)Z_{nt}:\; 0\leq t\leq t_0 \,\big|\, Z_n>0\big\} \quad \text{as } n\uparrow\infty.$$

We start with the description of the desired limiting process X^*. First we consider a continuous-state branching process $\big\{X(t):\,0\leq t<\infty\big\}$ of index $1+\alpha$; more precisely, X is a $[0,\infty)$-valued (time-homogeneous) Markov process with càdlàg paths and transition Laplace functions

$$\mathbf{E}\big\{\mathrm{e}^{-\lambda X(t)}\,\big|\,X(0)=x\big\} \;=\; \exp\big[-x\,(t+\lambda^{-\alpha})^{-1/\alpha}\big], \qquad \lambda,t,x\geq 0. \tag{73}$$

Introduce a random variable $\chi \geq 0$ having the Laplace transform

$$\mathbf{E}\mathrm{e}^{-\lambda\chi} \;=\; 1-(1+\lambda^{-\alpha})^{-1/\alpha}, \qquad \lambda\geq 0, \tag{74}$$

(see, e.g., [Sla68]). According to a general construction as in Durrett [Dur76], we introduce a Markov process $\{X^{+}(t) : 0 \le t \le 1\}$ with càdlàg paths and with the following properties: For $y > 0$ and $0 < t \le 1$,

$$\mathbf{P}\big(X^{+}(t) \in dy\big) = t^{-1/\alpha}\,\mathbf{P}(t^{1/\alpha}\chi \in dy)\,\mathbf{P}\big\{X(1-t) > 0 \,\big|\, X(0) = y\big\}, \tag{75}$$

and, for $x > 0$ and $0 \le s < t \le 1$,

$$\mathbf{P}\big\{X^{+}(t) \in dy \,\big|\, X^{+}(s) = x\big\} = \tag{76}$$
$$\frac{\mathbf{P}\big\{X(t-s) \in dy;\, X(t-s) > 0 \,\big|\, X(0) = x\big\}}{\mathbf{P}\big\{X(t-s) > 0 \,\big|\, X(0) = x\big\}}\,\mathbf{P}\big\{X(1-t) > 0 \,\big|\, X(0) = x\big\}.$$

Finally, we define the Markov process $\{X^{*}(t) : 0 \le t < \infty\}$ as a concatenation of processes X^{+} and X; more precisely,

$$X^{*}(t) := \begin{cases} X^{+}(t) & \text{if } 0 \le t \le 1, \\ X^{X^{+}(1)}(t-1) & \text{if } t \ge 1, \end{cases} \tag{77}$$

where X^{x} refers to X starting from $X(0) = x$, and this family $\{X^{x} : x > 0\}$ is chosen independently of $\{X^{+}(t) : 0 \le t \le 1\}$.

Proposition 13 (A conditional invariance principle) *Let* $0 < t_0 < \infty$. *The following convergence in law on* $D\,[0, t_0]$ *holds:*

$$\big\{Q(n)Z_{nt} :\ 0 \le t \le t_0 \,\big|\, Z_n > 0\big\} \underset{n\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \big\{X^{*}(t) :\ 0 \le t \le t_0\big\}. \tag{78}$$

Proof It suffices to show that for $x > 0$,

$$\begin{aligned} &\big\{Q(n)Z_{nt} :\ 0 \le t \le t_0 \,\big|\, Z_0 = x/Q(n)\big\} \\ &\underset{n\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \big\{X(t) :\ 0 \le t \le t_0 \,\big|\, X(0) = x\big\}, \end{aligned} \tag{79}$$

in $D[0, t_0]$, and that

$$\begin{aligned} &\big\{Q(n)Z_{nt} :\ 0 \le t \le 1 \,\big|\, Z_n > 0\big\} \\ &\underset{n\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \big\{X^{+}(t) :\ 0 \le t \le 1 \,\big|\, X^{+}(0) = 0\big\}, \end{aligned} \tag{80}$$

in $D[0, 1]$. In fact, from (79) and (80), the Markov properties of the processes X^{+} and X, as well as the definition of X^{*}, the statement (78) follows.

From the conditional limit theorem in [Sla68] it is easy to derive that for any $t, x > 0$,

$$\big\{Q(n)Z_{nt} \,\big|\, Z_0 = x/Q(n)\big\} \underset{n\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \big\{X(t) \,\big|\, X(0) = x\big\}. \tag{81}$$

By Theorem 3.4 in Grimvall [Gri74], the validity of (81) implies (79).

To demonstrate (80), we will use Theorem 3.9 from [Dur76] according to which it is necessary to show in our situation that, besides (79), the following four statements hold:

$$\mathbf{P}\Big\{\inf_{0\le s\le t} X(s)>0 \,\Big|\, X(0)=x\Big\}>0, \qquad t,x>0; \tag{82}$$

$$\mathbf{P}\big\{Z_{nt_n}>0 \,\big|\, Z_0=x_n/Q(n)\big\} \to \mathbf{P}\big\{X(t)>0 \,\big|\, X(0)=x\big\} \tag{83}$$

whenever $t_n \to t>0$ and $x_n \to x>0$;

$$\mathbf{P}\big\{Z_{nt_n}>0 \,\big|\, Z_0=x_n/Q(n)\big\} \to 0 \tag{84}$$

whenever $t_n \to t>0$ and $x_n \to 0$; finally,

$$X^+(t) \overset{\mathcal{L}}{\Longrightarrow} 0 \quad \text{as } t\downarrow 0. \tag{85}$$

Since the state 0 is absorbing for the branching process X, we have for $t,x>0$,

$$\begin{aligned}\mathbf{P}\Big\{\inf_{0\le s\le t} X(s)>0 \,\Big|\, X(0)=x\Big\} &= \mathbf{P}\big\{X(t)>0 \,\big|\, X(0)=x\big\} \\ = 1-\lim_{\lambda\downarrow 0}\mathbf{E}\big\{\mathrm{e}^{-\lambda X(t)} \,\big|\, X(0)=x\big\} &= 1-\exp[-xt^{-1/\alpha}],\end{aligned} \tag{86}$$

proving (82). As $n\uparrow\infty$, if $t_n\to t>0$ and $x_n\to x>0$, then, in view of (14) and properties of slowly varying functions,

$$\frac{Q(nt_n)}{Q(n)} \to t^{-1/\alpha}, \tag{87}$$

and therefore,

$$\begin{aligned}\mathbf{P}\big\{Z_{nt_n}>0 \,\big|\, Z_0=x_n/Q(n)\big\} &= 1-\big(1-Q(nt_n)\big)^{x_n/Q(n)} \\ &\to 1-\exp[-xt^{-1/\alpha}].\end{aligned} \tag{88}$$

Combining (86) and (88), we get (83) and (84).

Finally, it follows from (74) that $\mathbf{E}\chi^\beta<\infty$ for any $\beta\in(1,1+\alpha)$. Using this fact and (75), we see that for such β and $\varepsilon>0$,

$$\mathbf{P}\big(X^+(t)\ge\varepsilon\big) \le t^{-1/\alpha}\,\mathbf{P}(t^{1/\alpha}\chi\ge\varepsilon) \le t^{(\beta-1)/\alpha}\,\varepsilon^{-\beta}\,\mathbf{E}\chi^\beta \to 0 \tag{89}$$

as $t\downarrow 0$. This justifies (85). Thus, (80) is proved, and the proof of the lemma is complete. $\square$

2.4 On the limiting process X^*

For convenience, we introduce the notation

$$V^*(T) := \sup_{0 \le s \le T-1} \int_s^{s+1} X^*(u)du, \qquad T \ge 1, \tag{90}$$

and later we write $V(T)$ in case of working with X instead of X^*. In this subsection we establish the following result.

Proposition 14 (Exact velocity) *As* $T \uparrow \infty$,

$$\mathbf{E}V^*(T) \sim \log T. \tag{91}$$

The proof of this proposition will be prepared by the following three lemmas.

Lemma 15 (Two estimates) *For* $\beta \in (1, 1+\alpha)$,

$$\mathbf{P}\Big(\sup_{0 \le t \le 1} X^+(t) \ge x\Big) \le 1 \wedge \frac{c_{11}}{x^\beta}, \qquad x > 0. \tag{92}$$

Moreover, for all $T > 0$ *and* $0 < y \le x$,

$$\mathbf{P}\Big\{\sup_{0 \le t \le T} X(t) \ge x \,\Big|\, X(0) = y\Big\} \le c_{11} T^{(\beta-1)/\alpha} \frac{y}{x^\beta}. \tag{93}$$

Proof From the Donsker-Prokhorov invariance principle and (80) it follows that for $x > 0$,

$$\mathbf{P}\Big(\sup_{0 \le t \le 1} X^+(t) \ge x\Big) = \lim_{n \uparrow \infty} \mathbf{P}\big\{Q(n)M_n \ge x \,\big|\, Z_n > 0\big\}. \tag{94}$$

Using Doob's inequality and Lemma 11 we obtain

$$\begin{aligned} \mathbf{P}\big\{Q(n)M_n \ge x \,\big|\, Z_n > 0\big\} &\le Q^{-1}(n)\, \mathbf{P}\big(Q(n)M_n \ge x\big) \\ &\le Q^{\beta-1}(n) \frac{\mathbf{E}Z_n^\beta}{x^\beta} \le \frac{c_{11}}{x^\beta}. \end{aligned} \tag{95}$$

From here and (94), estimate (92) follows.

To prove (93) observe that by the Doob and von Bahr-Esseen inequalities and Lemma 11,

$$\begin{aligned} \mathbf{P}\big\{Q(n)M_{nT} \ge x \,\big|\, Z_0 = y/Q(n)\big\} &\le \frac{y}{Q(n)} \frac{Q^\beta(n)\, \mathbf{E}\big\{Z_{nT}^\beta \,\big|\, Z_0 = 1\big\}}{x^\beta} \\ &\le \frac{c_{11} y}{x^\beta} \Big(\frac{Q(n)}{Q(nT)}\Big)^{\beta-1}. \end{aligned} \tag{96}$$

On the other hand, by (79) and the latter estimate,

$$\mathbf{P}\Big\{\sup_{0\le t\le T} X(t)\ge x\;\Big|\;X(0)=y\Big\} \;=\; \lim_{n\uparrow\infty}\mathbf{P}\big\{Q(n)M_{nT}\ge x\,\big|\,Z_0=y/Q(n)\big\}$$

$$\le\; \frac{c_{11}y}{x^{\beta}}\lim_{n\uparrow\infty}\Big(\frac{Q(n)}{Q(nT)}\Big)^{\beta-1} \;=\; \frac{c_{11}y}{x^{\beta}}\,T^{(\beta-1)/\alpha}. \tag{97}$$

The first lemma is proved. □

Lemma 16 (Tail of maximum) *For* $0<y\le x<\infty$,

$$\mathbf{P}\Big\{\sup_{0\le t<\infty} X(t)\ge x\;\Big|\;X(0)=y\Big\} \;=\; 1-\Big(1-\frac{y}{x}\Big)^{\alpha}. \tag{98}$$

Proof This follows from the Donsker-Prokhorov invariance principle, Lemma 1 in [BV96], and Theorem 2 in [Pak78]. □

Lemma 17 (Minimal population) *For* $\varepsilon>0$ *there exists a constant* $c_{17}=c_{17}(\varepsilon)$, *such that for all* $\beta\in(1,1+\alpha)$ *and* $x>0$,

$$\mathbf{P}\Big\{\inf_{0\le t\le 1} X(t)\le x\;\Big|\;X(0)=(1+\varepsilon)\,x\Big\} \;\le\; 1\wedge\frac{c_{17}}{x^{\beta-1}}\,. \tag{99}$$

Proof Applying (66), we see that for $\varepsilon>0$, $j\ge 1$, and $y>0$ satisfying $\varepsilon y\ge 2$ the inequality

$$\mathbf{P}\Big\{\min_{l<j} Z_l<y\;\Big|\;Z_0=(1+\varepsilon)y\Big\} \;\le\; c_{12}\Big(\frac{1}{Q(j)y}\Big)^{\beta-1} \tag{100}$$

is true. Choosing now $y=x/Q(j)$, we get for all sufficiently large j,

$$\mathbf{P}\Big\{\min_{l<j} Z_l<x/Q(j)\;\Big|\;Z_0=(1+\varepsilon)x/Q(j)\Big\} \;\le\; \frac{c_{12}}{x^{\beta-1}}\,. \tag{101}$$

Hence, applying the Donsker-Prokhorov principle, (79), (83), and letting $j\uparrow\infty$, the desired estimate follows. □

Having those three lemmas, the *Proof of Proposition* 14 is now given by the following two lemmas.

Lemma 18 (Upper expectation estimate) *We have*

$$\limsup_{T\uparrow\infty}\frac{1}{\log T}\,\mathbf{E}V^{*}(T)\;\le\;1. \tag{102}$$

Proof Clearly,

$$V^{*}(T)\;\le\;\sup_{0\le s\le T} X^{*}(s). \tag{103}$$

From definition (77) of X^* it follows that for $x>0$,

$$\mathbf{P}\Big(\sup_{0\le s\le T} X^*(s)\ge x\Big) \;\le\; \mathbf{P}\Big(\sup_{0\le s\le 1} X^+(s)\ge x\Big) + \int_0^x \mathbf{P}\big(X^+(1)\in dy\big)\,\mathbf{P}\Big\{\sup_{0\le t\le T} X(t)\ge x \;\Big|\; X(0)=y\Big\}. \tag{104}$$

In view of (92),

$$\int_1^\infty \mathbf{P}\Big(\sup_{0\le s\le 1} X^+(s)\ge x\Big)dx \;<\; \infty. \tag{105}$$

Fix any $0<\varepsilon<1$. By (93) and (98) we get for $x>0$, decomposing $(0,x)$,

$$\begin{aligned}&\int_0^x \mathbf{P}\big(X^+(1)\in dy\big)\,\mathbf{P}\Big\{\sup_{0\le t\le T} X(t)\ge x \;\Big|\; X(0)=y\Big\} \qquad (106)\\ &\le \int_0^{\varepsilon x} \mathbf{P}\big(X^+(1)\in dy\big)\min\Big(\frac{c_{11}}{x^\beta}\,T^{(\beta-1)/\alpha}y,\, 1-\Big(1-\frac{y}{x}\Big)^\alpha\Big)\\ &\qquad + \;\mathbf{P}\big(X^+(1)\ge\varepsilon x\big).\end{aligned}$$

Noting that by the mean value theorem, for all $y\le\varepsilon x$,

$$1-\Big(1-\frac{y}{x}\Big)^\alpha \;\le\; \frac{\alpha y}{x}\Big(1-\frac{y}{x}\Big)^{\alpha-1} \;\le\; \alpha\,(1-\varepsilon)^{\alpha-1}\,\frac{y}{x}\,, \tag{107}$$

we have the bound

$$\begin{aligned}&\int_0^{\varepsilon x} \mathbf{P}\big(X^+(1)\in dy\big)\min\Big(\frac{c_{11}}{x^\beta}\,T^{(\beta-1)/\alpha}y,\, 1-\Big(1-\frac{y}{x}\Big)^\alpha\Big)\\ &\qquad\le\; \min\Big(\frac{c_{11}}{x^\beta}\,T^{(\beta-1)/\alpha},\;(1-\varepsilon)^{\alpha-1}\,\frac{\alpha}{x}\Big),\end{aligned} \tag{108}$$

since $\mathbf{E}X^+(1)=\mathbf{E}\chi=1$. Therefore, decomposing $(1,\infty)$,

$$\begin{aligned}&\int_1^\infty dx\int_0^x \mathbf{P}\big(X^+(1)\in dy\big)\,\mathbf{P}\Big\{\sup_{0\le t\le T} X(t)\ge x \;\Big|\; X(0)=y\Big\} \qquad (109)\\ &\le\; \alpha\,(1-\varepsilon)^{\alpha-1}\int_1^{T^{1/\alpha}}\frac{dx}{x}+c_{11}T^{(\beta-1)/\alpha}\int_{T^{1/\alpha}}^\infty\frac{dx}{x^\beta}\\ &\qquad+\int_1^\infty dx\,\mathbf{P}\big(X^+(1)\ge\varepsilon x\big) \;\le\; (1-\varepsilon)^{\alpha-1}\,\log T+c+\frac{1}{\varepsilon}\,,\end{aligned}$$

where the last term follows by substitution and again by $\mathbf{E}X^+(1)=1$. This implies the claim. □

Lemma 19 (Lower expectation estimate) *We have*

$$\liminf_{T\uparrow\infty}\frac{1}{\log T}\,\mathbf{E}V^*(T)\;\ge\;1. \tag{110}$$

Proof Recalling notation V introduced around (90), it is not difficult to check that for $T \geq 2$ and $x > 0$,

$$\mathbf{P}\big(V^*(T) \geq x\big) \;\geq\; \int_{(0,\infty)} \mathbf{P}\big(X^+(1) \in dy\big)\, \mathbf{P}\big\{V(T-1) \geq x \,\big|\, X(0) = y\big\}.$$

Fix $\varepsilon \in (0,1)$ and put $\rho := \inf\big\{u \geq 0 : \, X(u) \geq (1+\varepsilon)x\big\}$ [being equal to infinity if $\sup_{u \geq 0} X(u) < (1+\varepsilon)x$]. Clearly, by the strong Markov property and properties of continuous-state branching processes,

$$\begin{aligned}
&\mathbf{P}\big\{V(T-1) \geq x \,\big|\, X(0) = y\big\} \qquad (111)\\
&\geq \int_0^{T-2} \mathbf{P}\left\{V(T-1) \geq x, \;\rho \in dw \,\big|\, X(0) = y\right\}\\
&\geq \int_0^{T-2} \mathbf{P}\Big\{\int_w^{w+1} X(u)du \geq x, \;\rho \in dw \,\Big|\, X(0) = y\Big\}\\
&\geq \int_0^{T-2} \mathbf{P}\Big\{\inf_{w \leq u \leq w+1} X(u) \geq x, \;\rho \in dw \,\Big|\, X(0) = y\Big\}.
\end{aligned}$$

Using the strong Markov property at time ρ, the latter integral coincides with

$$\begin{aligned}
&\int_0^{T-2} \int_{(1+\varepsilon)x}^{\infty} \mathbf{P}\Big\{\rho \in dw, \;X(w) \in dz \,\Big|\, X(0) = y\Big\} \qquad (112)\\
&\qquad\qquad \times \mathbf{P}\Big\{\inf_{0 \leq u \leq 1} X(u) \geq x \,\Big|\, X(0) = z\Big\}\\
&\geq \; \mathbf{P}\Big\{\inf_{0 \leq u \leq 1} X(u) \geq x \,\Big|\, X(0) = (1+\varepsilon)x\Big\} \mathbf{P}\left\{\rho \leq T-2 \,\big|\, X(0) = y\right\}.
\end{aligned}$$

Applying Lemma 17 we have, for all $x \geq x_0(\varepsilon)$,

$$\begin{aligned}
&\mathbf{P}\big\{V(T-1) \geq x \,\big|\, X(0) = y\big\}\\
&\geq \; (1-\varepsilon)\; \mathbf{P}\Big\{\sup_{0 \leq t \leq T-2} X(t) \geq (1+\varepsilon)x \,\Big|\, X(0) = y\Big\}. \qquad (113)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbf{P}\Big\{\sup_{0 \leq t \leq T-2} X(t) \geq (1+\varepsilon)x \,\Big|\, X(0) = y\Big\} \qquad (114)\\
&\geq \mathbf{P}\Big\{\sup_{0 \leq t < \infty} X(t) \geq (1+\varepsilon)x \,\Big|\, X(0) = y\Big\} - \mathbf{P}\big\{X(T-2) > 0 \,\big|\, X(0) = y\big\}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\mathbf{P}\big(V^*(T) \geq x\big)\\
&\geq (1-\varepsilon) \int_0^{\infty} \mathbf{P}\big(X^+(1) \in dy\big)\, \mathbf{P}\Big\{\sup_{0 \leq t < \infty} X(t) \geq (1+\varepsilon)x \,\Big|\, X(0) = y\Big\}\\
&\qquad - \int_0^{\infty} \mathbf{P}\big(X^+(1) \in dy\big)\, \mathbf{P}\left\{X(T-2) > 0 \,\big|\, X(0) = y\right\}. \qquad (115)
\end{aligned}$$

From (74), (75), and (86), we see that

$$\begin{aligned}&\int_0^\infty \mathbf{P}\big(X^+(1)\in dy\big)\,\mathbf{P}\left\{X(T-2)>0\,\big|\,X(0)=y\right\}\\ &=\ 1-\int_0^\infty \mathbf{P}\big(X^+(1)\in dy\big)\exp\Big[-y(T-2)^{-1/\alpha}\Big]\ =\ (T-1)^{-1/\alpha}. \qquad (116)\end{aligned}$$

It follows from (98) that for fixed $y>0$,

$$\mathbf{P}\Big\{\sup_{0\le t<\infty} X(t)\ge x\ \Big|\ X(0)=y\Big\}\ \sim\ \frac{\alpha y}{x}\quad \text{as } x\uparrow\infty. \qquad (117)$$

Hence, by Fatou's lemma,

$$\begin{aligned}&\liminf_{x\uparrow\infty}\, x\int_0^\infty \mathbf{P}\Big\{\sup_{0\le t<\infty} X(t)\ge (1+\varepsilon)x\ \Big|\ X(0)=y\Big\}\,\mathbf{P}\big(X^+(1)\in dy\big)\\ &\ge\ \int_0^\infty \lim_{x\uparrow\infty} x\,\mathbf{P}\Big\{\sup_{0\le t<\infty} X(t)\ge (1+\varepsilon)x\ \Big|\ X(0)=y\Big\}\,\mathbf{P}\big(X^+(1)\in dy\big)\\ &=\ \alpha\,(1+\varepsilon)^{-1}\int_0^\infty y\,\mathbf{P}\big(X^+(1)\in dy\big)\ =\ \alpha\,(1+\varepsilon)^{-1}. \qquad (118)\end{aligned}$$

Substituting arrays (116) and (118) in (115) gives, for sufficiently large x,

$$\mathbf{P}\big(V^*(T)\ge x\big)\ \ge\ \frac{\alpha}{x}\,(1-2\varepsilon)-(T-1)^{-1/\alpha}. \qquad (119)$$

Hence, for sufficiently large T,

$$\int_{T^\varepsilon}^{T^{1/\alpha}} \mathbf{P}\big(V^*(T)\ge x\big)\,dx\ \ge\ (1-2\varepsilon)\,(1/\alpha-\varepsilon)\,\alpha\log T-(1-1/T)^{-1/\alpha}.$$

From here the statement of the lemma follows. □

3 Proof of the main results

3.1 Proof of Theorem 2

(a) By monotonicity in $j\ge 1$,

$$\mathbf{P}\big(M(j)\ge n\big)\ \le\ \mathbf{P}\big(M(\infty)\ge n\big). \qquad (120)$$

On the other hand,

$$\begin{aligned}\mathbf{P}\big(M(j)\ge n\big)\ &\ge\ \mathbf{P}\big(M(j)\ge n,\,Z_j=0\big)\ =\ \mathbf{P}\big(M(\infty)\ge n,\,Z_j=0\big)\\ &\ge\ \mathbf{P}\big(M(\infty)\ge n\big)-\mathbf{P}(Z_j>0). \qquad (121)\end{aligned}$$

Applying Lemmas 6 and 7(a) to (120) and (121) with $j=j_n$ justifies part **(a)** of the theorem.

(b) Recalling $M(\infty) = S_0(\infty)$, since

$$\begin{aligned} \mathbf{P}\big(M(j_n) \geq n,\, Z_{j_n} = 0\big) \;&=\; \mathbf{P}\big(M(\infty) \geq n,\, Z_{j_n} = 0\big) \\ &=\; \mathbf{P}\big(S_0(\infty) \geq n\big) - \mathbf{P}\big(S_0(\infty) \geq n,\, Z_{j_n} > 0\big), \end{aligned} \tag{122}$$

we have

$$\begin{aligned} \mathbf{P}\big(M(j_n) \geq n\big) \;=\; \mathbf{P}\big(M(j_n) \geq n,\, Z_{j_n} > 0\big) + \mathbf{P}\big(S_0(\infty) \geq n\big) \\ -\; \mathbf{P}\big(S_0(\infty) \geq n,\, Z_{j_n} > 0\big). \end{aligned} \tag{123}$$

We investigate each term at the right hand side of array (123) separately. First we deal with the second term. By (9), Lemma 7(b), and our conditions,

$$\begin{aligned} \mathbf{P}\big(S_0(\infty) \geq n\big) &\sim \mathbf{P}\Big(S_0(\infty) \geq \frac{j_n}{yQ(j_n)}\Big) \sim \mathbf{P}\Bigg(S_0(\infty) \geq \frac{\big(j_n y^{-\frac{\alpha}{1+\alpha}}\big)}{Q\big(j_n y^{-\frac{\alpha}{1+\alpha}}\big)}\Bigg) \\ &\sim \frac{\alpha^{\frac{1}{1+\alpha}}}{\Gamma\big(\frac{\alpha}{1+\alpha}\big)}\, Q\big(j_n y^{-\frac{\alpha}{1+\alpha}}\big) \sim \frac{(\alpha y)^{\frac{1}{1+\alpha}}}{\Gamma\big(\frac{\alpha}{1+\alpha}\big)}\, Q(j_n) \quad \text{as } n \uparrow \infty. \end{aligned} \tag{124}$$

To study the asymptotic behavior of the last probability in array (123), note that, for any fixed $T \geq 1$,

$$\begin{aligned} \mathbf{P}\big\{S_0(\infty) \geq n \,\big|\, Z_{j_n} > 0\big\} \;=\; \mathbf{P}\big\{S_0(\infty) \geq n,\, Z_{Tj_n} = 0 \,\big|\, Z_{j_n} > 0\big\} \\ +\; \mathbf{P}\big\{S_0(\infty) \geq n,\, Z_{Tj_n} > 0 \,\big|\, Z_{j_n} > 0\big\}. \end{aligned} \tag{125}$$

The first probability term at the right hand side of decomposition (125) can be estimated from above as follows:

$$\mathbf{P}\big\{S_0(Tj_n) \geq n,\, Z_{Tj_n} = 0 \,\big|Z_{j_n} > 0\big\} \;\leq\; \mathbf{P}\big\{S_0(Tj_n) \geq n \,\big|Z_{j_n} > 0\big\}. \tag{126}$$

Concerning the other probability term in decomposition (125), in view of (14) and properties of slowly varying functions there exists a constant $c_{(127)}$ such that for all $n \geq 1$ and $j_n \geq 1$,

$$\begin{aligned} \mathbf{P}\big\{S_0(\infty) \geq n,\, Z_{Tj_n} > 0 \,\big|\, Z_{j_n} > 0\big\} \;&\leq\; \mathbf{P}\big\{Z_{Tj_n} > 0 \,\big|\, Z_{j_n} > 0\big\} \\ &=\; \frac{Q\,(Tj_n)}{Q\,(j_n)} \;\leq\; \frac{c_{(127)}}{T^{1/\alpha}}. \end{aligned} \tag{127}$$

Combining (125) − (127),

$$\begin{aligned} 0 \leq\; \mathbf{P}\big\{S_0(\infty) \geq n \,\big|\, Z_{j_n} > 0\big\} - \mathbf{P}\big\{S_0(Tj_n) \geq n \,\big|\, Z_{j_n} > 0\big\} \\ \leq\; c_{(127)} T^{-1/\alpha}. \end{aligned} \tag{128}$$

Using the Donsker-Prokhorov invariance principle and Proposition 13 we see that

$$\begin{aligned} &\lim_{n\uparrow\infty} \mathbf{P}\big\{S_0(Tj_n) \geq n \;\big|\, Z_{j_n} > 0\big\} \\ &=\; \lim_{n\uparrow\infty} \mathbf{P}\bigg\{ \int_0^{T-j_n^{-1}} Q(j_n) Z_{vj_n} dv \;\geq\; \frac{n\,Q(j_n)}{j_n} \;\bigg|\; Z_{j_n} > 0\bigg\} \\ &=\; \mathbf{P}\Big(\int_0^T X^*(v) dv \geq y^{-1}\Big). \end{aligned} \tag{129}$$

Since T can be made arbitrary large, (128) and (129) imply

$$\lim_{n\uparrow\infty} \mathbf{P}\big\{S_0(\infty) \geq n \,\big|\, Z_{j_n} > 0\big\} \;=\; \mathbf{P}\Big(\int_0^\infty X^*(v)dv \geq y^{-1}\Big). \tag{130}$$

Thus, as $n \uparrow \infty$,

$$\mathbf{P}\big(S_0(\infty) \geq n,\, Z_{j_n} > 0\big) \;\sim\; Q(j_n)\,\mathbf{P}\Big(\int_0^\infty X^*(v)dv \geq y^{-1}\Big). \tag{131}$$

Finally, to deal with the first probability term at the right-hand side of array (123), observe that

$$\begin{aligned}\mathbf{P}\big\{M(j_n) \geq n \,\big|\, Z_{j_n} > 0\big\} \;=\; &\mathbf{P}\big\{M(j_n) \geq n,\, Z_{Tj_n} = 0 \,\big|\, Z_{j_n} > 0\big\} \\ &+\; \mathbf{P}\big\{M(j_n) \geq n,\, Z_{Tj_n} > 0 \,\big|\, Z_{j_n} > 0\big\}.\end{aligned} \tag{132}$$

Here the first probability term can be written as

$$\mathbf{P}\big\{M_{Tj_n}(j_n) \geq n,\, Z_{Tj_n} = 0 \,\big|\, Z_{j_n} > 0\big\} \;\leq\; \mathbf{P}\big\{M_{Tj_n}(j_n) \geq n \,\big|\, Z_{j_n} > 0\big\},$$

whereas for the other term we have the upper bound

$$\mathbf{P}\big\{Z_{Tj_n} > 0 \,\big|\, Z_{j_n} > 0\big\}. \tag{133}$$

Both together and applying again (127) gives

$$\begin{aligned}0 \;\leq\; \mathbf{P}\big\{M(j_n) \geq n \,\big|\, Z_{j_n} > 0\big\} - \mathbf{P}\big\{M_{Tj_n}(j_n) \geq n \,\big|\, Z_{j_n} > 0\big\} \\ \leq\; c\,T^{-1/\alpha}.\end{aligned} \tag{134}$$

Using the representation

$$M_{Tj_n}(j_n) = \max_{0\leq k\leq (T-1)j_n} \sum_{l=k}^{k+j_n-1} Z_l \;=\; j_n \max_{0\leq u\leq T-1} \int_u^{u+1-j_n^{-1}} Z_{vj_n}dv \tag{135}$$

and applying the Donsker-Prokhorov invariance principle as well as Proposition 13 once again, we see that $j_n^{-1}Q(j_n)n \to y$ implies

$$\begin{aligned}&\lim_{n\uparrow\infty} \mathbf{P}\big\{M_{Tj_n}(j_n) \geq n \,\big|\, Z_{j_n} > 0\big\} \\ &= \lim_{n\uparrow\infty} \mathbf{P}\Big\{j_n^{-1}Q(j_n)M(j_n) \geq \frac{Q(j_n)n}{j_n}\Big| Z_{j_n} > 0\Big\} \;=\; \mathbf{P}\big(V^*(T) \geq y^{-1}\big).\end{aligned} \tag{136}$$

Hence, letting $T \to \infty$ and taking into account (134) we obtain

$$\lim_{n\uparrow\infty} \mathbf{P}\big\{M(j_n) \geq n \,\big|\, Z_{j_n} > 0\big\} \;=\; \mathbf{P}\big(V^*(\infty) \geq y^{-1}\big). \tag{137}$$

Combining (137), (124), and (131) we see that $Q(j_n)nj_n^{-1} \to y \in (0,\infty)$ implies

$$\mathbf{P}\big(M(j_n) \geq n\big) \;\sim\; \psi\,(y)\,Q(j_n), \tag{138}$$

where

$$\psi(y) := \mathbf{P}\big(V^*(\infty) \geq y^{-1}\big) + \frac{(\alpha y)^{\frac{1}{1+\alpha}}}{\Gamma\big(\frac{\alpha}{1+\alpha}\big)} - \mathbf{P}\Big(\int_0^\infty X^*(v)dv \geq y^{-1}\Big). \tag{139}$$

Note that $\psi(y) > 0$ since the first term at the right-hand side of array (123) is of order $Q(j_n)$ while the difference of the second and third terms is non-negative.

(c) To establish (11) observe that $M(j) \leq jM(1)$ and therefore by Theorem 1 from [BV96], for any $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that for n and j satisfying $nj^{-1} > K$,

$$\mathbf{P}\big(M(j) \geq n\big) \leq \mathbf{P}\big(M(1) \geq nj^{-1}\big) \leq \frac{\alpha(1+\varepsilon)j}{n}. \tag{140}$$

To get a similar estimate from below note that for $\varepsilon \in (0,1)$,

$$\begin{aligned}\mathbf{P}\big(M(j) \geq n\big) &\geq \mathbf{P}\big(M(j) \geq n,\, M(1) \geq (1+\varepsilon)nj^{-1}\big) \qquad (141)\\ &= \sum_{l=1}^{\infty} \mathbf{P}\big(M(j) \geq n,\ \varrho = l\big),\end{aligned}$$

where $\varrho := \min\big\{l : Z_l \geq (1+\varepsilon)nj^{-1}\big\}$ is the first moment when the generation size exceeds $(1+\varepsilon)nj^{-1}$. By the Markov property we get

$$\begin{aligned}\mathbf{P}\big(M(j) \geq n,\ \varrho = l\big) &= \sum_{r \geq (1+\varepsilon)nj^{-1}} \mathbf{P}\big(M(j) \geq n,\, Z_l = r,\ \varrho = l\big)\\ &\geq \sum_{r \geq (1+\varepsilon)nj^{-1}} \mathbf{P}\Big(\sum_{i=l}^{l+j-1} Z_{i+l} \geq n,\, Z_l = r,\ \varrho = l\Big)\\ &= \sum_{r \geq (1+\varepsilon)nj^{-1}} \mathbf{P}\Big\{\sum_{i=0}^{j-1} Z_i \geq n \,\Big|\, Z_0 = r\Big\}\mathbf{P}(Z_l = r,\ \varrho = l)\\ &\geq \mathbf{P}(\varrho = l)\,\mathbf{P}\Big\{\sum_{i=0}^{j-1} Z_i \geq n \,\Big|\, Z_0 = (1+\varepsilon)nj^{-1}\Big\}. \qquad (142)\end{aligned}$$

Therefore,

$$\mathbf{P}\big(M(j) \geq n\big) \geq \mathbf{P}\big(M(1) \geq (1+\varepsilon)nj^{-1}\big)\ \mathbf{P}\Big\{\sum_{l=0}^{j-1} Z_l \geq n \,\Big|\, Z_0 = (1+\varepsilon)nj^{-1}\Big\}. \tag{143}$$

Choose $\beta \in (1, 1+\alpha)$ and using (67), we obtain for $nj^{-1} \geq 2/\varepsilon$,

$$\mathbf{P}\big(M(j) \geq n\big) \geq \mathbf{P}\big(M(1) \geq (1+\varepsilon)nj^{-1}\big)\Big(1 - c_{12}\Big(\frac{j}{n\,Q(j)}\Big)^{\beta-1}\Big). \tag{144}$$

Observing that $nj_n^{-1} \to \infty$ by our assumption in (c) and recalling that $\mathbf{P}\big(M(1) \geq x\big) \sim \alpha/x$ as $x \uparrow \infty,$ estimates (140) and (144) together imply

$$\mathbf{P}\big(M(j_n) \geq n\big) \;\sim\; \mathbf{P}\big(M(1) \geq nj_n^{-1}\big) \;\sim\; \frac{\alpha j_n}{n} \quad \text{as} \quad \frac{j_n}{n\,Q(j_n)} \to 0. \tag{145}$$

Theorem 2 is proved. □

3.2 Proof of Theorem 1(a)

Since $M_m(j) \leq M(j)$ for all $j, m \geq 1$, it follows from (120) that

$$\mathbf{P}\big(M_m(j) \geq n\big) \;\leq\; \mathbf{P}\big(M(j) \geq n\big) \;\leq\; \mathbf{P}\big(M(\infty) \geq n\big). \tag{146}$$

From here, (19), and properties of regularly varying functions we conclude that, for any $\varepsilon \in (0,1)$ and sufficiently large j,

$$\begin{aligned}\sum_{1 \leq n \leq \frac{j}{\varepsilon Q(j)}} \mathbf{P}\big(M_m(j) \geq n\big) \;&\leq\; \sum_{1 \leq n \leq \frac{j}{\varepsilon Q(j)}} \mathbf{P}\big(M(\infty) \geq n\big) \\ &\leq\; 2\,\frac{(1+\alpha)}{\alpha}\,\frac{j}{\varepsilon\,Q(j)}\,\mathbf{P}\Big(M(\infty) \geq \frac{j}{\varepsilon\,Q(j)}\Big).\end{aligned} \tag{147}$$

By Lemma 7(b), we have for $\varepsilon \in (0,1)$ and for sufficiently large j,

$$\mathbf{P}\Big(M(\infty) \geq \frac{j}{\varepsilon\,Q(j)}\Big) \;\leq\; \mathbf{P}\Big(M(\infty) \geq \frac{j}{Q(j)}\Big) \;\leq\; c\,Q(j). \tag{148}$$

Hence,

$$\sum_{1 \leq n \leq \frac{j}{\varepsilon Q(j)}} \mathbf{P}\big(M_m(j) \geq n\big) \;\leq\; \frac{c}{\varepsilon}\,j\,. \tag{149}$$

Moreover, for any $\beta \in (1, 1+\alpha)$,

$$\mathbf{P}\big(M_m(1) \geq x\big) \;\leq\; \frac{\mathbf{E}\big\{Z_m^{\beta} \,\big|\, Z_0 = 1\big\}}{x^{\beta}} \;\leq\; \frac{c\,Q^{1-\beta}(m)}{x^{\beta}}\,, \tag{150}$$

which, in view of $\mathbf{P}\big(M_m(j) \geq n\big) \leq \mathbf{P}\big(M_m(1) \geq nj^{-1}\big)$ implies

$$\begin{aligned}\sum_{n \,\geq\, \varepsilon\, \frac{j}{Q(m)}} \mathbf{P}\big(M_m(j) \geq n\big) \;&\leq\; \sum_{n \,\geq\, \varepsilon\, \frac{j}{Q(m)}} \mathbf{P}\big(M_m(1) \geq nj^{-1}\big) \\ &\leq\; c\,j^{\beta} Q^{1-\beta}(m) \sum_{n \,\geq\, \varepsilon\, \frac{j}{Q(m)}} n^{-\beta} \;\leq\; c\,\varepsilon^{1-\beta} j\end{aligned} \tag{151}$$

for $j \geq j_0$. Clearly,

$$\mathbf{P}\big(M(j) \geq n\big) - \mathbf{P}(Z_m > 0) \;\leq\; \mathbf{P}\big(M_m(j) \geq n\big) \;\leq\; \mathbf{P}\big(M(j) \geq n\big). \tag{152}$$

This and (145) show that for any $\delta \in (0,1)$ there exists an $\varepsilon \in (0,1)$ such that

$$(1-\delta)\,\frac{\alpha j}{n} - \mathbf{P}(Z_m > 0) \;\le\; \mathbf{P}\big(M_m(j) \ge n\big) \;\le\; (1+\delta)\,\frac{\alpha j}{n} \tag{153}$$

for all $n \ge \varepsilon^{-1} j/Q(j)$. Denoting

$$D_\varepsilon(j,m) \;:=\; \big\{n:\, \varepsilon^{-1} j/Q(j) \,\le\, n \,\le\, \varepsilon j/Q(m)\big\}\,, \tag{154}$$

we conclude that

$$\begin{aligned}(1-\delta)\,\alpha j \sum_{n\in D_\varepsilon(j,m)} 1/n - \varepsilon j \;&\le \sum_{n\in D_\varepsilon(j,m)} \mathbf{P}\big(M_m(j)\ge n\big)\\ &\le\; (1+\delta)\,\alpha j \sum_{n\in D_\varepsilon(j,m)} 1/n,\end{aligned} \tag{155}$$

that is,

$$\begin{aligned}(1-\delta)\,\alpha j\,\log\frac{Q(j)}{Q(m)} - c\,j \;&\le \sum_{n\in D_\varepsilon(j,m)} \mathbf{P}\big(M_m(j)\ge n\big)\\ &\le\; (1+\delta)\,\alpha j\,\log\frac{Q(j)}{Q(m)} \;+\; c\,j.\end{aligned} \tag{156}$$

Since the function ℓ_4 from (14) is slowly varying, there exists an $a>0$ and functions σ and θ satisfying $\sigma(x)\to\sigma\in(0,\infty)$ and $\theta(x)\to 0$ as $x\uparrow\infty$, such that (see [Sen76, Section 1.5])

$$\ell_4(n) \;=\; \sigma(n)\exp\Big[\int_a^n \frac{\theta(x)}{x}\,\mathrm{d}x\Big]. \tag{157}$$

Hence, it follows easily that for any $\mu>0$, there exists $w=w(\mu)$ such that

$$\Big(\frac{m}{j}\Big)^{-\mu/\alpha} \;\le\; \frac{\ell_4(j)}{\ell_4(m)} \;\le\; \Big(\frac{m}{j}\Big)^{\mu/\alpha} \quad \text{as } j/m \le w. \tag{158}$$

Therefore, for $j/m < w$,

$$\begin{aligned}(1-\delta)\,(1-\mu)\,j\,\log\frac{m}{j} \;&\le \sum_{n\in D_\varepsilon(j,m)} \mathbf{P}\big(M_m(j)\ge n\big)\\ &\le\; (1+\delta)\,(1+\mu)\,j\,\log\frac{m}{j}\,.\end{aligned} \tag{159}$$

Combining (149) – (159) and taking into account that δ and μ can be made arbitrarily small, we get

$$\mathbf{E}M_m(j) \;\sim\; j\,\log\frac{m}{j} \quad \text{as } j/m\to 0, \tag{160}$$

completing the proof of Theorem 1(a). $\square$

3.3 Proof of Theorem 1(b)

Clearly,

$$\begin{aligned}\mathbf{E}M_{Tj}(j) &= \mathbf{E}\big\{M_{Tj}(j);\, Z_j=0\big\} + \mathbf{E}\big\{M_{Tj}(j);\, Z_j>0\big\} \\ &= \mathbf{E}\big\{S_0(j);\, Z_j=0\big\} + \mathbf{E}\big\{M_{Tj}(j);\, Z_j>0\big\} \qquad (161)\end{aligned}$$

since

$$\begin{aligned}M_{Tj}(j)1_{\{Z_j=0\}} &= \max_{0\le k\le Tj-j} \sum_{l=k}^{k+j-1} Z_l 1_{\{Z_j=0\}} \qquad (162)\\ &= \max_{0\le k\le Tj-j} \sum_{l=k}^{j-1} Z_l 1_{\{Z_j=0\}} = \sum_{l=0}^{j-1} Z_l 1_{\{Z_j=0\}} = S_0(j)1_{\{Z_j=0\}}\,.\end{aligned}$$

We study each term in (161) separately, namely in Lemmas 20 and 22 below.

Lemma 20 (Restricted expectation asymptotics) *As* $j\uparrow\infty$,

$$a_j := \mathbf{E}\big\{S_0(j);\, Z_j=0\big\} \sim \frac{\alpha j}{2\alpha+1}\,. \qquad (163)$$

Remark 21 (Finite variance case) Under $\mathbf{V}\mathrm{ar}\xi<\infty$, this result was obtained by Karpenko and Nagaev in [KN93]. $\diamond$

Proof Set

$$h_j(s_1,s_2) := \mathbf{E}\Big\{s_1^{Z_j} s_2^{S_0(j)}\ \Big|\ Z_0=1\Big\}, \quad h_0(s_1,s_2) := s_1 s_2\,. \qquad (164)$$

Clearly,

$$\begin{aligned}h_j(s_1,s_2) &= \mathbf{E}\Big\{\mathbf{E}\Big\{s_1^{Z_j} s_2^{S_0(j)}\ \Big|\ Z_1\Big\}\ \Big|\ Z_0=1\Big\} \qquad (165)\\ &= \mathbf{E}\Big\{s_2\Big(\mathbf{E}\big\{s_1^{Z_{j-1}} s_2^{S_0(j-1)}\,\big|\, Z_0=1\big\}\Big)^{Z_1}\ \Big|\ Z_0=1\Big\} = s_2 f\big(h_{j-1}(s_1,s_2)\big).\end{aligned}$$

Hence,

$$\mathbf{E}\Big\{s_2^{S_0(j)},\, Z_j=0\ \Big|\ Z_0=1\Big\} = h_j(0,s_2) = s_2 f\big(h_{j-1}(0,s_2)\big). \qquad (166)$$

Note, that $h_j(0,1)=f_j(0)=\mathbf{P}\big\{Z_j=0\,\big|\,Z_0=1\big\}$ and $a_1=f_1(0)$. Differentiating (166) at $s_2=1-$ gives, for $j\ge 2$,

$$\begin{aligned}a_j &= f_j(0)+f'\big(f_{j-1}(0)\big)a_{j-1} \qquad (167)\\ &= f_j(0)+f'\big(f_{j-1}(0)\big)f_{j-1}(0)+f'\big(f_{j-1}(0)\big)f'\big(f_{j-2}(0)\big)a_{j-2}\end{aligned}$$

leading to

$$a_j = f_j(0)+\sum_{k=1}^{j-1} f_k(0)\prod_{r=k}^{j-1} f'\big(f_r(0)\big) = f_j(0)+d_j\sum_{k=1}^{j-1} f_k(0)\,\frac{1}{d_k}\,, \qquad (168)$$

where the d_k are as in Lemma 5. Recalling Lemma 5 and observing that $f_k(0) \uparrow 1$ as $k \uparrow \infty$, we get

$$\sum_{k=1}^{j-1} f_k(0)\,\frac{1}{d_k} \;\sim\; \sum_{k=1}^{j-1} \frac{k^{1+1/\alpha}}{l_3(k)} \;\sim\; \frac{\alpha}{2\alpha+1}\,\frac{j^{2+1/\alpha}}{l_3(j)} \quad \text{as } j \uparrow \infty. \tag{169}$$

From here, (168), and (15), the statement of the lemma follows easily. $\square$

Lemma 22 (Conditional expectation limit) *For* $T \geq 1$,

$$\lim_{j\to\infty} \mathbf{E}\big\{ j^{-1} Q(j)\, M_{Tj}(j) \,\big|\, Z_j > 0 \big\} \;=\; \mathbf{E} V^*(T). \tag{170}$$

Proof It follows from Proposition 13 and the Donsker-Prokhorov invariance principle that

$$\big\{ j^{-1} Q(j)\, M_{Tj}(j) \,\big|\, Z_j > 0 \big\} \underset{j\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} V^*(T). \tag{171}$$

To prove that convergence of the expectations takes place recall that

$$M_{Tj}(j) \;\leq\; M_{Tj}(Tj) \;=\; \sum_{l=0}^{Tj-1} Z_l\,. \tag{172}$$

Hence,

$$\mathbf{P}\big\{ j^{-1} Q(j)\, M_{Tj}(j) > y \,\big|\, Z_j > 0 \big\} \;\leq\; \mathbf{P}\big\{ j^{-1} Q(j)\, M_{Tj}(Tj) > y \,\big|\, Z_j > 0 \big\}$$

$$\leq\; \mathbf{P}\Big\{ Q(j) \max_{0\leq k\leq Tj} Z_k > y \;\Big|\; Z_j > 0 \Big\} \;\leq\; \frac{Q^\beta(j)}{Q(j)}\,\frac{\mathbf{E} Z_{Tj}^\beta}{y^\beta}\,, \tag{173}$$

the last step by Doob's inequality. By Lemma 11 and (14), we can continue with

$$\leq\; \frac{c}{y^\beta}\,\frac{Q^{\beta-1}(j)}{Q^{\beta-1}(Tj)} \;\leq\; \frac{c}{y^\beta}\, T^{(\beta-1)/\alpha}. \tag{174}$$

Therefore,

$$\mathbf{P}\big(V^*(T) > y \big) \;\leq\; \frac{c}{y^\beta}\, T^{(\beta-1)/\alpha}. \tag{175}$$

In order to complete the proof, note that since $\beta > 1$, derived chain of estimates (173) – (174) and inequality (175) provide the uniform integrability of the prelimiting and limiting variables in (171). Hence, claimed convergence (170) of moments follows. $\square$

Now we are ready *to complete the Proof of Theorem* 1. Clearly,

$$\begin{aligned} & j_m^{-1}\, \mathbf{E} M_m(j_m) && (176) \\ &= j_m^{-1}\, \mathbf{E}\{ M_m(j_m),\, Z_{j_m} = 0 \} + j_m^{-1}\, \mathbf{E}\{ M_m(j_m),\, Z_{j_m} > 0 \} \\ &= j_m^{-1}\, \mathbf{E}\{ S_0(j_m),\, Z_{j_m} = 0 \} + \mathbf{E}\big\{ j_m^{-1} Q(j_m)\, M_m(j_m) \,\big|\, Z_{j_m} > 0 \big\}. \end{aligned}$$

Applying Lemmas 20 and 22 with $T = 1/\eta$, we obtain

$$\lim_{m\uparrow\infty} j_m^{-1}\,\mathbf{E}M_m(j_m) \;=\; \frac{\alpha}{2\alpha+1} + \mathbf{E}V^*(1/\eta) \;=:\; \varphi(\eta), \tag{177}$$

that is (7). Recalling Proposition 14, we see that (8) is valid as well. □

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